Proofs

(Direct Proof, Proof by contrapositive)

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DISCRETE STRUCTURES

Lecture 06

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Definitions

- A **theorem** is a statement that can be shown to be true. In mathematical writing, the term theorem is usually reserved for a statement that is considered at least somewhat important.
- Less important theorems sometimes are called **propositions**. (Theorems can also be referred to as **facts** or **results**.) A theorem may be the universal quantification of a conditional statement with one or more premises and a conclusion.
- However, it may be some other type of logical statement, as the examples later in this chapter will show. We demonstrate that a theorem is true with a **proof**.

Methods of Proving Theorems

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- Direct Proof
- Proof by Contraposition (Contrapositive)

Methods of Proving Theorems cont… Direct Proof

- The simplest (from a logic perspective) style of proof is a ***direct proof***.
- Often all that is required to prove something is a systematic explanation of what everything means.
- Direct proofs are especially useful when proving implications.
- The general format to prove p $\rightarrow$ q is this:
- Assume p. Explain, explain, …, explain. Therefore q.

Methods of Proving Theorems cont…
Direct Proof

- Often we want to prove universal statements, perhaps of the form $\forall x(p(x) \rightarrow q(x))$.
- Again, we will want to assume p(x) is true and deduce q(x).
- But what about the x? We want this to work for all x.
- We accomplish this by fixing x to be an arbitrary element (of the sort we are interested in).

Methods of Proving Theorems cont… Direct Proof (Example)

- **Prove:** For all integers n, if n is even, then n^2 is even.
- The format of the proof with be this: Let n be an arbitrary integer. Assume that n is even. Therefore n^2 is even.
- To fill in the details, we will basically just explain what it means for n to be even, and then see what that means for n^2. Here is a complete proof.

Methods of Proving Theorems cont…
Direct Proof (Example)

- The integer **n** is even if there exists an integer **k** such that **n = 2k**, and **n** is odd if there exists an integer **k** such that **n = 2k + 1**. (every integer is either even or odd, and no integer is both even and odd.)
- Two integers have the same parity when both are even or both are odd; they have opposite parity when one is even and the other is odd.

Methods of Proving Theorems cont…
Direct Proof (Example)

- **Proof** Let n be an arbitrary integer.
- Suppose n is even.
- Then $n=2k$ for some integer k.
- Now $n^2=(2k)^2=4k^2=2(2k^2)$. Since $2k^2$ is an integer, n^2 is even.
- Hence proved (For all integers n, if n is even, then n^2 is even.)

Methods of Proving Theorems cont… Direct Proof (Example 2)

Give a direct proof that if **m** and **n** are both perfect squares, then **nm** is also a perfect square.

- (An integer **a** is a perfect square if there is an integer **b** such that $a=b^2$.)

Methods of Proving Theorems cont… Direct Proof (Example 2)

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- **<u>Solution:</u>** To produce a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, i.e. **m** and **n** are both perfect squares.
- By the definition of a perfect square, it follows that there are integers **s** and **t** such that $\mathbf{m = s^2}$ and $\mathbf{n = t^2}$.

Methods of Proving Theorems cont…
Direct Proof (Example 2)

- The goal of the proof is to show that **mn** must also be a perfect square when **m** and **n** are; looking ahead we see how we can show this by substituting $\mathbf{s^2}$ for **m** and $\mathbf{t^2}$ for **n** into **mn**.
- This tells us that $\mathbf{mn = s^2t^2}$.
- Hence, $\mathbf{mn = s^2t^2} = \mathbf{(ss)(tt)} = \mathbf{(st)(st)} = \mathbf{(st)^2}$, using commutativity and associativity of multiplication.

Methods of Proving Theorems cont… Direct Proof (Example 2)

- By the definition of perfect square, it follows that **mn** is also a perfect square, because it is the square of **st**, which is an integer.
- We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Methods of Proving Theorems cont...
Proof by Contrapositive

- Recall that an implication $p \rightarrow q$ is logically equivalent to its contrapositive $\neg q \rightarrow \neg p$.
- There are plenty of examples of statements which are hard to prove directly, but whose contrapositive can easily be proved directly.
- This is all that proof by contrapositive does.
- It gives a direct proof of the contrapositive of the implication.

Methods of Proving Theorems cont…
Proof by Contrapositive

- This is enough because the contrapositive is logically equivalent to the original implication.
- The skeleton of the proof of $p \to q$ by contrapositive will always look roughly like this: Assume $\neg q$. Explain, explain, … explain. Therefore $\neg p$.
- As before, if there are variables and quantifiers, we set them to be arbitrary elements of our domain. Here are a couple examples:

Methods of Proving Theorems cont… Proof by Contrapositive (Example)

- Prove the statement "for all integers n, if n^2 is even, then n is even" true?
- **Solution:** This is the converse of the statement we proved above using a direct proof.
- A direct proof of this statement would require fixing an arbitrary n and assuming that n^2 is even. But it is not at all clear how this would allow us to conclude anything about n. Just because $n^2=2k$ does not in itself suggest how we could write n as a multiple of 2.

Methods of Proving Theorems cont… Proof by Contrapositive (Example)

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- Try something else: write the contrapositive of the statement. We get, for all integers n, if n is odd then n^2 is odd. Our proof will look something like this:
- Let n be an arbitrary integer. Suppose that n is not even. …. Therefore n^2 is not even.

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Methods of Proving Theorems cont… Proof by Contrapositive (Example)

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- Now we fill in the details:
- **Proof:** We will prove the contrapositive. Let n be an arbitrary integer. Suppose that n is not even, and thus odd. Then $n=2k+1$ for some integer k.
- Now $n^2=(2k+1)^2=4k^2+4k+1=2(2k^2+2k)+1$.
- Since $2k^2+2k$ is an integer, we see that n^2 is odd and therefore not even.